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INTERNATIONAL JOURNAL OF ENGINEERING SCIENCES & RESEARCH TECHNOLOGY

Characterize the Intra-regular and the Regular Ordered Semi-groups in Terms of Fuzzy Sets

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Abstract

In mathematics, fuzzy sets are sets whose elements have degrees of membership. Fuzzy sets were introduced by Lotfi A. Zadeh and Dieter Klaua in 1965 as an extension of the classical notion of set. At the same time, Salii (1965) defined a more general kind of structures called L-relations, which he studied in an abstract algebraic context. In this paper we characterize the intra-regular, the left (right) regular and the completely regular ordered semi-groups in terms of fuzzy sets.

Keywords: Fuzzy Set Theory, Ordered Semi-group; Intra-regular and completely regular Fuzzy Set.

Introduction and Prerequisites on Terms of Fuzzy Sets

In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition an element either belongs or does not belong to the set. By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the real unit interval [0, 1]. Fuzzy sets generalize classical sets, since the indicator functions of classical sets are special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1. In fuzzy set theory, classical bivalent sets are usually called *crisp* sets. The fuzzy set theory can be used in a wide range of domains in which information is incomplete or imprecise, such as bioinformatics [1-2].

Let $(S; ., ; \leq)$ be an ordered semigroup (: an ordered set with a multiplication which is compatible with the ordering). For a subset H of S, $(H]$ denotes the subset of S defined by $(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}$. For $H = \{a; b; c; ......\}$ we write $(a; b; c; .....]$ instead of $(\{a; b; c; .....\}]$. Clearly $S = (S]$, and for any subsets $A; B$ of S, we have $A \subseteq (A]$, if $A \subseteq B$ then $(A] \subseteq (B]$, $(A](B] \subseteq (AB]$, and $((A]] = (A]$. A nonempty subset A of S is called a *left* (resp. *right*) *ideal* of S if (1) $SA \subseteq A$ (resp. $AS \subseteq A$) and (2) if $a \in A$ and $S \ni b \leq a$, then $b \in A$, that is $(A] = A$. A is called an *ideal* of S if it is both a left and a right ideal of S. A nonempty subset A of S is called a *bi-ideal* of S if (1) $ASA \subseteq A$ and 2) if $a \in A$ and $S \ni b \leq a$, then $b \in A$. For an element a of S, $L(a)$, $R(a)$, $I(a)$, $B(a)$ denote the left ideal, right ideal, the ideal and the bi-ideal of S, respectively, generated by a, and we have

$L(a) = (a \cup Sa]$, $R(a) = (a \cup aS]$, $I(a) = (a \cup aS \cup Sa \cup aSa]$,
$B(a) = (a \cup aSa]$.

A left (resp. right) ideal A of S is clearly a subsemigroup of S i.e. $A2 \subseteq A$. A nonempty subset A of S is called an *interior ideal* of S if

1) $SAS \subseteq A$ and

2) if $a \in A$ and $S \ni b \leq a$, then $b \in A$. If A an ideal of S, then A is an interior ideal of S. Indeed, $S(AS) \subseteq SA \subseteq A$ and $(A] = A$. If A is a left (resp. right) ideal of S, then A is a bi-ideal of S. An ordered semigroup $(S; ., ; \leq)$ is called *left regular* if for every $a \in S$ there exists $x \in S$ such that $a \leq xa2$, that is $a \in (Sa2]$ for all $a \in S$ or $A \subseteq (SA2]$ for all $A \subseteq S$. It is called *right regular* if for every $a \in S$ there exists $x \in S$ such that $a \leq a2x$, that is $a \in (a2S]$ for all $a \in S$ or $A \subseteq (A2S]$ for all $A \subseteq S$. An ordered semigroup $(S; ., ; \leq)$ is called *regular* if for any $a \in S$ there exists $x \in S$ such that $a \leq axa$ i.e. $a \in (aSa]$ for every $a \in S$ or $A \subseteq (ASA]$ for every $A \subseteq S$. It is called *intra-regular* if for each $a \in S$ there exist $x; y \in S$ such that $a \leq xa2y$, that is $a \in (Sa2S]$ for all $a \in S$ or $A \subseteq (SA2S]$ for all $A \subseteq S$. An ordered semigroup $(S; ., ; \leq)$ is called *completely regular* if it is at the same time left regular, right regular and regular. In regular and in intra-regular ordered semi-groups the ideals and the interior ideals are the same. A subset T of S is called *semiprime* if for any $a \in S$ such that $a2 \in T$, we have $a \in T$, equivalently, if $A \subseteq S$ such that $A2 \subseteq T$ implies $A \subseteq T$.

If S is an ordered semigroup, a *fuzzy subset of S* (or *fuzzy set in S*) is a mapping f of S into the real closed interval $[0; 1]$ of real numbers. For a subset A of S, denote by fA the characteristic function on A, that is the fuzzy subset of S defined by

$$fA : S \rightarrow [0; 1] \,|\, fA(x) := \begin{cases} 1 \text{ if } x \in A \\ \\ 0 \text{ if } x \neq \in A \end{cases}$$

A fuzzy subset f of S is called a *fuzzy subsemigroup* of S if
(1) $f(xy) \geq \min\{f(x), f(y)\}$ for every $x; y \in S$ and
(2) if $x \leq y$, then $f(x) \geq f(y)$.

A fuzzy subset f of S is called a *fuzzy left ideal* (resp. *fuzzy right ideal*) of S if
(1) $f(xy) \geq f(y)$ (resp. $f(xy) \geq f(x)$) for every $x; y \in S$ and
(2) if $x \leq y$, then $f(x) \geq f(y)$.

For a fuzzy subset f of S, we say that f is a *fuzzy ideal* of S if it is both a fuzzy left and a fuzzy right ideal of S. A fuzzy subset f of S is called a *fuzzy bi-ideal* of S if
(1) $f(xyz) \geq \min\{f(x); f(z)\}$ for all $x; y; z \in S$ and
(2) if $x \leq y$, then $f(x) \geq f(y)$.
It is called a *fuzzy interior ideal* of S if
(1) $f(xay) \geq f(a)$ for all $x; a; y \in S$ and
(2) if $x \leq y$, then $f(x) \geq f(y)$.

Intra-regular ordered semi-groups play an important role is studying the structure of ordered semi-groups. An ordered semigroup S is intra-regular if and only if it is a semilattice of simple semi-groups, equivalently, if S is a union of simple subsemi-groups of S [2], which means that intra-regular ordered semi-groups are decomposable into simple components. Moreover, an ordered semigroup S is both regular and intra-regular if and only if it is a semilattice of simple and regular semi-groups]. Recall that every completely regular ordered semigroup is, by definition, left (resp. right) regular. Every left (resp. right) regular ordered semigroup is intra-regular. It is known, by the same author, that the intra-regular, left regular, and completely regular ordered semi-groups can be also defined as the ordered semi-groups in which the ideals, the left ideals, and the bi-ideals, respectively, are semiprime [3-5] and [8]. In this paper adds some additional information on the same type of ordered semi-groups using fuzzy sets as well.

Main Results
Lemma 1. *Let S be an ordered semigroup. If A is a left (resp right) ideal of S, then the characteristic function fA is a fuzzy left (resp. fuzzy right) ideal of S. "Conversely", if A is a nonempty subset of S and fA a fuzzy left (resp. fuzzy right) ideal of S, then A is a left (resp. right) ideal of S.*

Lemma 2. *Let S be an ordered semigroup. If A is a bi-ideal of S, then the characteristic function fA is a fuzzy bi-ideal of S. "Conversely", if A is a nonempty subset of S and fA a fuzzy bi-ideal of S, then A is a bi-ideal of S.*

Lemma 3. *Let S be an ordered semigroup. If A is a subsemigroup of S, then the characteristic function fA is a fuzzy subsemigroup of S. "Conversely", if A is a nonempty subset of S and fA a fuzzy subsemigroup of S, then A is subsemigroup of S.*

Lemma 4. (cf. also [9; Proposition 2.3]) *Let S is an ordered semigroup. If A is an interior ideal of S, then the characteristic function fA is a fuzzy interior ideal of S. "Conversely", if A is a nonempty subset of S and fA a fuzzy interior ideal of S, then A is an interior ideal of S.*

Definition 5. Let S be an ordered semigroup. A fuzzy subset f of S is called *fuzzy semiprime* if
$f(a) \geq f(a2)$ for every $a \in S$:

Theorem 6. *Let S be an ordered semigroup. The following are equivalent:* (1) *S is intra-regular.*
(2) *Every interior ideal of S is semiprime.*
(3) *Every fuzzy interior ideal of S is fuzzy semiprime.*
(4) *If f is a fuzzy interior ideal and at the same time a fuzzy subsemigroup of S, then f(a) = f(a2) for every a ∈ S.*
(5) $a \in I(a^2)$ *for every a ∈ S.*
(6) $I(a) = I(a^2)$ *for every a ∈ S.*
(7) *Every ideal of S is semiprime.*
(8) *Every fuzzy ideal of S is fuzzy semiprime.*

Proof. (1) ⇒ (2). Let A be an interior ideal of S and $a \in S$, $a^2 \in A$. Since S is intra-regular, we have $a \in (Sa^2S] \subseteq (SAS] \subseteq (A] = A$, and A is semiprime.

(2) =⇒ (3). Let f be a fuzzy interior ideal of S and $a \in S$. The set $(Sa2S]$ is an interior ideal of S. This is because it is a nonempty subset of S,
$S(Sa^2S]S = (S](Sa^2S](S] \subseteq (S(Sa^2S)S] \subseteq (Sa^2S];$ and $((Sa^2S)] = (Sa^2S]$. Then, by (2), $(Sa^2S]$ is semiprime. Since $a^4 \in (Sa^2S]$, we have
$a2 \in (Sa^2S]$, and $a \in (Sa^2S]$.
Then $a \leq xa^2y$ for some $x; y \in S$. Since f is a fuzzy interior ideal of S, we have $f(a) \geq f(xa^2y) \geq f(a^2)$, so f is fuzzy semiprime.

(3) =⇒ (4). Let f be a fuzzy interior ideal at the same time a fuzzy subsemigroup of S and $a \in S$. By (3), f is a fuzzy semiprime fuzzy subsemigroup of S, so we have
$f(a) \geq f(a^2) \geq \min\{f(a); f(a)\} = f(a);$ then $f(a) = f(a^2)$.

(4) =⇒ (5). Let $a \in S$. Since $I(a^2)$ is an ideal of S, $I(a^2)$ is an interior ideal at the same time a subsemigroup of S. By Lemmas 3 and 4, the characteristic function $fI(a^2)$ is a fuzzy interior ideal and at the same time a fuzzy subsemigroup of S. By (4), we have
$fI(a^2)(a) = fI(a^2)(a^2) = 1$. Then $a \in I(a^2)$.

(5) =⇒ (6). If $a \in S$ then, since $a \in I(a^2)$, we have
$I(a) \subseteq I(a^2) = (a^2 \cup Sa^2 \cup a^2S \cup Sa^2S] \subseteq (Sa \cup aS \cup SaS] \subseteq I(a);$
thus we have $I(a) = I(a^2)$.
(6) =⇒ (7). If A is an ideal of S and $a \in S$ such that $a^2 \in A$ then, by (6), we have
$a \in I(a) = I(a^2) \subseteq A$, So A is semiprime.

(7) =⇒ (8). Let f be a fuzzy ideal of S and $a \in S$. As we have already seen the set $(Sa^2S]$ is a nonempty subset of S and $((Sa^2S]] = (Sa^2S]$, moreover $S(Sa^2S] = (S](Sa^2S] \subseteq (S^2a^2S] \subseteq (SaS]$, so $(Sa^2S]$ is a left ideal of S, similarly it is a right ideal, and so an ideal of S. By hypothesis, $(Sa^2S]$ is semiprime. Since $a^4 \in (Sa^2S]$, we have
$a^2 \in (Sa^2S]$, and
$a \in (SaS]$. Then $a \leq xa^2y$ for some $x; y \in S$. Then we get $f(a) \geq f(xa^2y) \geq f(a^2)$, so f is fuzzy semiprime.

(8) =⇒ (7). Let A be an ideal of S and $a \in S$, $a^2 \in A$. Since fA is a fuzzy ideal of S, by hypothesis, fA is fuzzy semiprime, so we have $fA(a) \geq fA(a^2) = 1$. On the other hand, since fA is a fuzzy subset of S, we have $fA(a) \leq 1$. Thus we have $fA(a) = 1$, and $a \in A$, so A is semiprime.

(7) =⇒ (1). Let $a \in S$. Since $(Sa2S]$ is an ideal of S, by hypothesis, it is semiprime. Since $a^4 \in (Sa^2S]$, we have $a \in (Sa^2S]$, so S is intra-regular. One can also prove directly the implication (8) ⇒ (1). In this case the proof is more technical.

Theorem 7. *Let S be an ordered semigroup. The following are equivalent:*
(1) *S is left regular.*
(2) *The left ideals of S are semiprime.*
(3) *The fuzzy left ideals of S are fuzzy semiprime.*
(4) *If f is a fuzzy left ideal and at the same time a fuzzy subsemigroup of S, then*
$f(a) = f(a^2)$ *for every a ∈ S.*
(5) $a \in L(a^2)$ *for every a ∈ S.*
(6) $L(a) = L(a^2)$ *for every a ∈ S.*

Proof. (1) $\Rightarrow$ (2). Let A be a left ideal of S and $a \in S$ such that $a^2 \in A$. Then $a \in (Sa^2] \subseteq (SA] \subseteq (A] = A$, so A is semiprime.

(2) $\Rightarrow$ (3). Let f be a fuzzy left ideal of S and $a \in S$. The set $(Sa^2]$ is a left ideal of S. Indeed, $\varnothing \neq (Sa^2] \subseteq S$, $S(Sa^2] = (S](Sa^2] \subseteq (S(Sa^2)] \subseteq (Sa^2]$, and $((Sa^2]] = (Sa^2]$. By (2), $(Sa^2]$ is semiprime. Since $a^4 \in (Sa^2]$, we have $a^2 \in (Sa^2]$, and $a \in (Sa^2]$. Then $a \le xa^2$ for some $x \in S$ from which $f(a) \ge f(xa^2) \ge f(a^2)$, and f is fuzzy semiprime.

(3) $\Rightarrow$ (4). Let f be a fuzzy left ideal at the same time a fuzzy subsemigroup of S and $a \in S$. Since f is a fuzzy left ideal of S, by (3), f is fuzzy semiprime which means $f(a) \ge f(a^2)$. Since f is a fuzzy subsemigroup of S, we have $f(a^2) \ge \min\{f(a); f(a)\}$ $= f(a)$. Then we obtain $f(a) = f(a^2)$.

(4) $\Rightarrow$ (5). Let $a \in S$. As $L(a^2)$ is a left ideal and a subsemigroup of S, by Lemmas 1 and 3, $fL(a^2)$ is a fuzzy left ideal and a fuzzy subsemigroup of S. By (4), we have $fL(a^2)(a) = fL(a^2)(a^2) = 1$. Then $a \in L(a^2)$ and condition (5) is satisfied.

(5) $\Rightarrow$ (6). If $a \in S$, then $a \in L(a^2) = (a^2 \cup Sa^2] \subseteq (Sa] \subseteq L(a)$. Then we have $L(a) \subseteq L(a^2) \subseteq L(a)$, and $L(a) = L(a^2)$.

(6) $\Rightarrow$ (1). Let $a \in S$. We have $a \in L(a) = L(a^2) = (a^2 \cup Sa^2]$. Then $a2 \in (a^2 \cup Sa^2](a] \subseteq (a^3 \cup Sa^3] \subseteq (Sa^2]$: Thus we have $a \in ((Sa^2] \cup Sa^2] = ((Sa^2]] = (Sa^2]$, and S is left regular.
The right analogue of Theorem 7 also holds and we have the following theorem.

Theorem 8. *Let S be an ordered semigroup. The following are equivalent:*
(1) *S is right regular.*
(2) *The right ideals of S are semiprime.*
(3) *The fuzzy right ideals of S are fuzzy semiprime.*
(4) *If f is a fuzzy right ideal and at the same time a fuzzy subsemigroup of S, then*
$f(a) = f(a2)$ *for every $a \in S$.*
(5) $a \in R(a2)$ *for every $a \in S$.*
(6) $R(a) = R(a2)$ *for every $a \in S$.*

Lemma 9. *An ordered semigroup S is completely regular if and only if, for every $a \in S$, we have $a \in (a^2Sa^2)$*

Proof. $\Rightarrow$. Since S is completely regular, we have
$a \in (aSa] \subseteq ((a^2S]S(Sa^2]] = ((a^2S](S](Sa^2]] \subseteq ((a^2S)S(Sa^2)] \subseteq (a^2Sa^2]$;
thus $a \in (a^2Sa^2]$.
Let $a \in S$. Since $a \in (a^2Sa^2] \subseteq (aSa]$; $(Sa^2]$; $(a^2S]$, S is regular, left regular and right regular.

Theorem 10. *Let S be an ordered semigroup. The following are equivalent:*
(1) *S is completely regular.*
(2) *Every bi-ideal of S is semiprime.*
(3) *Every fuzzy bi-ideal of S is fuzzy semiprime.*
(4) $a \in B(a2)$ *for every $a \in S$.*
(5) $B(a) = B(a2)$ *for every $a \in S$.*

Proof. (1) $\Rightarrow$ (2). Let A be a bi-ideal of S and $a \in S$ such that $a^2 \in A$. Since S is completely regular, by Lemma 9, we have $a \in (a^2Sa^2] \subseteq (ASA] \subseteq (A] = A$, and $a \in A$.

(2) $\Rightarrow$ (3). Let f be a fuzzy bi-ideal of S and $a \in S$. The set $(a^2Sa^2]$ is a bi-ideal of S.
This is because $(a^2Sa^2]$ is a nonempty subset of S,
$(a^2Sa^2]S(a^2Sa^2] = (a^2Sa^2](S](a^2Sa^2] \subseteq ((a^2Sa^2)S(a^2Sa^2)] \subseteq (a^2Sa^2]$;
and $((a^2Sa^2]] = (a^2Sa^2]$. By (2), $(a^2Sa^2]$ is semiprime. Since $(a^4)^2 = a^8 \in (a^2Sa^2]$, we have $(a^2)^2 = a^4 \in (a^2Sa^2]$, $a^2 \in (a^2Sa^2]$, and $a \in (a^2Sa^2]$. Then $a \le a^2xa^2$ for some $x \in S$. Since f is a fuzzy bi-ideal of S, we have
$f(a) \ge f(a^2xa^2) \ge \min\{f(a^2); f(a^2)\} = f(a^2)$; and f is fuzzy semiprime.

(3) $\Longrightarrow$ (4). Let $a \in S$. We consider the bi-ideal $B(a^2)$ of S generated by a. This is the set $(a2 \cup a^2Sa^2]$. Since $B(a^2)$ is a bi-ideal of S, by Lemma 2, $fB(a^2)$ is a fuzzy bi-ideal of S. By (3), we have $fB(a^2)(a) \geq fB(a^2)(a^2) = 1$. Since $fB(a2)$ is a fuzzy set in S, we have
$fB(a^2)(a) \leq 1$. Then $fB(a^2)(a) = 1$, and $a \in B(a^2)$.

(4) $\Longrightarrow$ (5). Let $a \in S$. Since $B(a)$ is the bi-ideal of S generated by a, by (4), we have
$a \in B(a) \subseteq B(a^2) = (a^2 \cup a^2Sa^2]$. Then
$a^2 \in (a^2 \cup a^2Sa^2](a] \subseteq ((a^2 \cup a^2Sa^2)a] = (a^3 \cup a^2Sa^3] \subseteq (aSa] \subseteq (a \cup aSa] = B(a);$
and $B(a^2) \subseteq B(a)$. Thus we obtain $B(a) = B(a^2)$.

(5) $\Longrightarrow$ (1). Let $a \in S$. By (5), we have $a \in B(a) = B(a^2) = (a^2 \cup a^2Sa^2]$. Then $a \leq a^2$ or $a \leq a^2ya^2$ for some $y \in S$. If $a \leq a^2$,
then $a \leq a^2a^2 = aaa^2 \leq a^2aa^2$. For $x := a$, we get $a \leq a^2xa^2$, and the proof is complete.
A left (or right) regular ordered semigroup S is intra-regular. Indeed, if S is left regular, then for each $a \in S$, we have
$a \in (Sa^2] \subseteq (S(Sa^2]a] \subseteq ((S](Sa^2](a]] \subseteq (S(Sa^2)a] \subseteq (Sa^2S];$ So S is intra-regular.

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